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Feynman integrals with tensorial structure in the negative dimensional integration scheme

A.T. Suzuki, A.G.M. Schmidt

Universidade Estadual Paulista – Instituto de Física Teórica, R.Pamplona, 145, São Paulo SP CEP 01405-900, Brazil

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Abstract. The negative-dimensional integration method (NDIM) is revealing itself as a very useful technique for computing massless and/or massive Feynman integrals, covariant and noncovariant alike. Up until now, however, the illustrative calculations done using such method have been mostly covariant scalar integrals, without numerator factors. We show here how those integrals with tensorial structures also can be handled straightforwardly and easily. However, contrary to the absence of significant features in the usual approach, here the NDIM also allows us to come across surprising unsuspected bonuses. Toward this end, we present two alternative ways of working out the integrals and illustrate them by taking the easiest Feynman integrals in this category that emerge in the computation of a standard one-loop self-energy diagram. One of the novel and heretofore unsuspected bonuses is that there are degeneracies in the way one can express the final result for the referred Feynman integral.

1 Introduction

In an effort to make sense of diverging integrals that have emerged in the field- theoretical approach to transition amplitudes, scattering matrices, and so forth, physicists introduced and developed the concept of extended dimensions[1]. This can be interpreted in a pragmatic way as a mere artifact of getting around a difficult problem. Nonetheless, the principle of analytic continuation behind it is mathematically well-founded, being self-consistent and well-defined. Thus, if we say that the beauty and the power of mathematics reside in the possibility of defining abstract entities that have no real connection to our physical world, and that from such entities, we can draw pertinent and meaningful properties that become relevant to our dimensionality, then the effort to go to these frontiers is worthwhile and enriching. We think of the negative-dimensional integration method (NDIM) in such terms.

We work with negative dimensions and with precise analytic continuations, so that interesting results emerge from our exploration. A very useful technique stemming from this excursion is the method of integrating Feynman integrals in negative dimensions [2]. Instead of having the usual field propagators in the denominator of the integrands, we have them here as numerators. In other words, what we have here are, in essence, integrands of polynomial type. Of course, once the integral is performed in negative dimensions, it must be analytically continued back to our real, positive-dimensional world. The basis for doing this is set forth in our previous papers [3–5].

Our aim in this work is to further illustrate the methodology of the NDIM, and for this purpose we take examples from the one-loop vacuum-polarization tensor diagram, which generates some Feynman integrals with tensorial structures. We show that the calculation of integrals with tensorial structures can be dealt with properly using the NDIM technology. Moreover, we show that this can be approached in at least two ways, which we consider with details in the next sections. A first approach is to just "copy" the steps used in the traditional positive-dimensional approach, i.e., using derivative identities in the integrands. A second, novel approach is to define right from the beginning the relevant negative-dimensional integral corresponding to the Feynman integral we want to evaluate and proceed from there. Just to make the illustrations simpler and clearer, we restrict ourselves to massless fields, but the generalization to massive ones is not difficult to do.

2 Using differential identities

Let us first consider the following (vectorial) Feynman integral:

$$I^\mu = \int \mathrm{d}^{2\omega} q \, \frac{q^\mu}{q^2\,(q-p)^2}\,, \tag{1}$$

which clearly emerges in the calculation of a vacuum-polarization tensor of, e.g., quantum electrodynamics. This, of course, is easily calculated in the standard procedure of positive dimensions. The next question we address is: How is this done in the NDIM context?

The structure of the above integral immediately suggests that a possible way of starting off NDIM calculation is to consider a Gaussian-like integral of the type

$$\mathcal{G}^\mu = \int \mathrm{d}^{2D} q \, q^\mu \, \mathrm{e}^{-\alpha q^2 - \beta (q-p)^2}\,. \tag{2}$$

This, in terms of the negative-dimensional integral $\mathcal{I}^\mu(i,j,D;p)$ is therefore given by

$$\mathcal{G}^\mu = \sum_{i,j=0}^{\infty} (-1)^{i+j} \frac{\alpha^i\,\beta^j}{i!\,j!} \int \mathrm{d}^{2D}q\, q^\mu\, (q^2)^i\, [(q-p)^2]^j$$
$$= \sum_{i,j=0}^{\infty} (-1)^{i+j} \frac{\alpha^i\,\beta^j}{i!\,j!}\, \mathcal{I}^\mu(i,j,D;p)\,. \qquad (3)$$

On the other hand, performing the momentum integration of equation (2) through the use of the following identity,

$$q^\mu\, \mathrm{e}^{2\beta q\cdot p} = \frac{1}{2\beta}\frac{\partial}{\partial p_\mu}\, \mathrm{e}^{2\beta q\cdot p}\,, \qquad (4)$$

we get

$$\mathcal{G}^\mu = \frac{\beta}{\lambda} p^\mu \left(\frac{\pi}{\lambda}\right)^D \exp\left(-\frac{\alpha\,\beta}{\lambda}\,p^2\right)$$
$$= p^\mu \pi^D \sum_{x,a,b=0}^{\infty} (-1)^x\,(-x-D-1)!\,\frac{\alpha^{x+a}\,\beta^{x+b+1}}{a!\,b!}$$
$$\times \frac{(p^2)^x}{x!}\delta_{a+b,x+D+1}\,, \qquad (5)$$

where $\lambda = \alpha+\beta$.

Comparison of equations (3) and (5), term by term, yields the result for $\mathcal{I}^\mu(i,j,D;p)$. After an analytic continuation to positive dimensions and negative values of exponents $(i,\, j)$ [4], we get

$$I^\mu = \mathcal{I}^\mu_{AC}$$
$$= \pi^D\, p^\mu\, (p^2)^\sigma\, \frac{(-i|\sigma)\,(-j|\sigma+1)}{(-\sigma|2\sigma+D+1)}\,. \qquad (6)$$

where we have used $\sigma = i+j+D$ and the definition for Pocchhammer's symbols

$$(-i|\sigma) \equiv (-i)_\sigma = \frac{\Gamma(-i+\sigma)}{\Gamma(\sigma)}\,. \qquad (7)$$

Next, we consider the tensorial Feynman integral

$$I^{\mu\nu} = \int \mathrm{d}^{2\omega}q\, \frac{q^\mu\, q^\nu}{q^2\,(q-p)^2}\,. \qquad (8)$$

The procedure is completely analogous, now starting from

$$\mathcal{G}^{\mu\nu} = \int \mathrm{d}^{2D}q\, q^\mu\, q^\nu\, \mathrm{e}^{-\alpha q^2-\beta(q-p)^2}\,. \qquad (9)$$

Here we quote only the final result, which reads

$$I^{\mu\nu} = \mathcal{I}^{\mu\nu}_{AC}$$
$$= \pi^D\,(p^2)^\sigma \left\{ p^\mu\, p^\nu\, \frac{(-i|\sigma)\,(-j|\sigma+2)}{(-\sigma|2\sigma+D+2)} \right.$$
$$\left. -\frac{g^{\mu\nu}\,p^2}{2}\,\frac{(-i|\sigma+1)\,(-j|\sigma+1)}{(-\sigma-1|2\sigma+D+3)} \right\}\,. \qquad (10)$$

In a similar manner, we can evaluate the following integrals very easily:

$$I^{\mu\nu\rho} = \mathcal{I}^{\mu\nu\rho}_{AC}$$
$$= \pi^D\,(p^2)^\sigma \left\{ \frac{p^2\,T^{\mu\nu\rho}}{2}\frac{(-i|\sigma+1)\,(-j|\sigma+2)}{(-\sigma-1|2\sigma+D+4)} \right.$$
$$\left. +p^\mu\,p^\nu\,p^\rho\,\frac{(-i|\sigma)\,(-j|\sigma+3)}{(-\sigma|2\sigma+D+3)} \right\}, \qquad (11)$$

where $T^{\mu\nu\rho} = p^\mu\, g^{\nu\rho} + p^\nu\, g^{\mu\rho} + p^\rho\, g^{\mu\nu}$, and

$$I^{\mu\nu\rho\varsigma} = \mathcal{I}^{\mu\nu\rho\varsigma}_{AC}$$
$$= \pi^D\,(p^2)^\sigma \left\{ \frac{p^4\,\mathcal{A}^{\mu\nu\rho\varsigma}}{4}\,\mathbf{\Gamma}_{\mathcal{A}} \right.$$
$$\left. +\frac{p^2\,\mathcal{B}^{\mu\nu\rho\varsigma}}{2}\,\mathbf{\Gamma}_{\mathcal{B}} + p^\mu p^\nu p^\rho p^\varsigma\,\mathbf{\Gamma}_{\mathcal{P}} \right\}, \qquad (12)$$

where

$$\mathbf{\Gamma}_{\mathcal{A}} \equiv \frac{(-i|\sigma+2)\,(-j|\sigma+2)}{(-\sigma-2|2\sigma+D+6)}$$
$$\mathbf{\Gamma}_{\mathcal{B}} \equiv \frac{(-i|\sigma+1)\,(-j|\sigma+3)}{(-\sigma-1|2\sigma+D+5)}$$
$$\mathbf{\Gamma}_{\mathcal{P}} \equiv \frac{(-i|\sigma)\,(-j|\sigma+4)}{(-\sigma|2\sigma+D+4)} \qquad (13)$$

with $\mathcal{A}^{\mu\nu\rho\varsigma} = g^{\mu\nu}\,g^{\rho\varsigma} + g^{\mu\rho}\,g^{\nu\varsigma} + g^{\mu\varsigma}\,g^{\nu\rho}$ and $\mathcal{B}^{\mu\nu\rho\varsigma} = p^\mu p^\nu g^{\rho\varsigma} +$permutations. All these results agree with those given in Appendix A of [6].

3 Using pure NDIM technique

In order to calculate tensorial structures in Feynman integrals, we can adopt another alternative approach. Let us consider the following integral:

$$J = \int \mathrm{d}^{2D}q \frac{(2q\cdot p)^l}{q^2\,(q-p)^2}\,, \qquad l \geq 0 \qquad (14)$$

Of course, for $l > 0$, the tensorial structure is implicit, being contracted with external vector p. The advantage of this approach is that it takes care of all the tensorial structures at the same time.

So, instead of using, e.g., (2) or (9) as our starting point, we see from the structures of the Feynman integrals in (1) and (8) another possible way of defining the Gaussian-like integral of interest to begin with, namely,

$$\mathcal{H} = \int \mathrm{d}^{2D}q\, \mathrm{e}^{-\alpha q^2-\beta(q-p)^2-\gamma(2q\cdot p)}\,. \qquad (15)$$

This defines the negative-dimensional integral $\mathcal{J}(i,j,l,D;p)$ as follows:

$$\mathcal{H} = \sum_{i,j,l=0}^{\infty} (-1)^{i+j+l}\,\frac{\alpha^i\,\beta^j\,\gamma^l}{i!\,j!\,l!}$$
$$\times \int \mathrm{d}^{2D}q\,(q^2)^i\,[(q-p)^2]^j\,(2q\cdot p)^l$$
$$= \sum_{i,j,l=0}^{\infty} (-1)^{i+j+l}\,\frac{\alpha^i\,\beta^j\,\gamma^l}{i!\,j!\,l!}\,\mathcal{J}(i,j,l,D;p)\,. \qquad (16)$$

On the other hand, from (15) we get also

$$\mathcal{H} = \pi^D \sum_{\substack{x,\ldots,b=0 \\ a+b=-\sigma'-D}}^{\infty} (-1)^{x+y}\, 2^y\, \frac{(-\sigma'-D)!\,(p^2)^{\sigma'}}{a!\,b!\,x!\,y!\,z!} \times \alpha^{x+a}\beta^{x+y+b}\gamma^{y+2z} \qquad (17)$$

where $\sigma' \equiv x+y+z = i+j+l+D$, or $\sigma' = \sigma + l$. Therefore, the solution for $\mathcal{J}(i,j,l,D;p)$ is obtained from the solution of a system of linear algebraic equations of the following form [4]:

$$\begin{cases} i = x+a \\ j = x+y+b \\ l = y+2z \\ \sigma' = x+y+z \end{cases} \qquad (18)$$

It is very easy to see that the above system is formed by four equations but five unknowns (the sum indices x,y,z,a,b). Therefore, it can be solved only in terms of one of the sum indices x, y, z, a, or b. For each of these remaining indices, the sum yields ${}_3F_2$ hypergeometric functions of unit argument, as follows:

$$\mathcal{J}^{AC}_{\{S\}} = \mathbf{\Lambda}_{\{S\}}\, {}_3F_2(\mathrm{a},\,\mathrm{b},\,\mathrm{c}\,;\mathrm{e},\,\mathrm{f}\,|\,1\,) \qquad (19)$$

where the set $\{S\} = \{x, y_{\rm even}, y_{\rm odd}, z, a, b\}$[1], with

$$\mathbf{\Lambda}_x = \pi^D\,(2p^2)^l\,(-4p^2)^{i+j+D} \times \frac{(-j|\,2i+2j+l+2D)}{(i+j+D|\,l+D)\,(1+l|\,2i+2j+2D)}, \qquad (20)$$

$$\mathbf{\Lambda}_y^{\rm even} = \pi^D\,(p^2)^{\sigma'} \times \frac{(-i|\,2i+\frac{1}{2}l+D)\,(-j|\,2j+\frac{1}{2}l+D)}{(-i-j-\frac{1}{2}l-D|\,2i+2j+\frac{3}{2}l+3D)\,(1+l|\,-\frac{1}{2}l)}, \qquad (21)$$

$$\mathbf{\Lambda}_y^{\rm odd} = -2\,\mathbf{\Lambda}_y^{\rm even} \times \frac{(i+\frac{1}{2}l+D|\,\frac{1}{2})\,(-i-j-\frac{1}{2}l-D|\,\frac{1}{2})\,(-\frac{1}{2}l|\,\frac{1}{2})}{(1-j-\frac{1}{2}l-D|\,\frac{1}{2})}, \qquad (22)$$

$$\mathbf{\Lambda}_z = \pi^D\,(2p^2)^l\,(p^2)^{i+j+D} \times \frac{(-i|\,2i+l+D)\,(-j|\,2j+D)}{(-i-j-D|\,2i+2j+l+3D)}, \qquad (23)$$

$$\mathbf{\Lambda}_a = \pi^D\,(2p^2)^l\,(p^2)^{i+j+D}\,(-4)^{j+D} \times \frac{(-j|\,2j+D)}{(1+l|\,2j+2D)}, \qquad (24)$$

$$\mathbf{\Lambda}_b = \frac{\pi^D\,(p^2)^{\sigma'}\,(-1)^l}{2^{2i+l+2D}} \times \frac{(-i|\,-i-j-l-2D)\,(-j|\,2i+j+l+2D)}{(1+l|\,i+D)}, \qquad (25)$$

[1] Note that the y index has been split into its even and odd sectors.

and the corresponding parameters of hypergeometric functions given by:

Parameters	${}_3F_2^x$	${}_3F_2^{y,\rm even}$	${}_3F_2^{y,\rm odd}$
a	$-i$	$i+\frac{1}{2}l+D$	$i+\frac{1}{2}l+D+\frac{1}{2}$
b	$-i-j-\frac{1}{2}l-D$	$-i-j-\frac{1}{2}l-D$	$-i-j-\frac{1}{2}l-D+\frac{1}{2}$
c	$-i-j-\frac{1}{2}l-D+\frac{1}{2}$	$-\frac{1}{2}l$	$-\frac{1}{2}l+\frac{1}{2}$
e	$1-i-j-D$	$1-j-\frac{1}{2}l-D$	$1-j-\frac{1}{2}l-D+\frac{1}{2}$
f	$1-2i-j-l-2D$	$\frac{1}{2}$	$\frac{3}{2}$

Parameters	${}_3F_2^z$	${}_3F_2^a$	${}_3F_2^b$
a	$j+D$	$-i$	$i+j+l+2D$
b	$-\frac{1}{2}l$	$i+j+l+2D$	$i+\frac{1}{2}l+D$
c	$-\frac{1}{2}l+\frac{1}{2}$	$j+D$	$i+\frac{1}{2}l+D+\frac{1}{2}$
e	$1+i+j+D$	$j+\frac{1}{2}l+D+\frac{1}{2}$	$1+i+l+D$
f	$1-i-l-D$	$1+j+\frac{1}{2}l+D$	$1+2i+j+l+2D$

Observe that in the process of analytic continuation to our physical world ($D>0$), exponents i, j are analytically continued to allow for *negative* values, whereas the exponent l must be left untouched, since, by definition, $l \geq 0$ in the original Feynman integral [7].

One of the interesting features of the NDIM technique is that it can give rise to degenerate solutions for the same Feynman integral. All the answers we have above, although seemingly distinct, are in fact only different ways of expressing the same thing. This means that for this particular case, where the solutions are degenerate, taking one of them will suffice. The equivalence of the different forms in which the solutions are expressed is shown in the appendix.

Given that we have this freedom of choice, we can look at the hypergeometric functions whose parameters are listed in the table and see that the most convenient solution is given by the one coming from solving the system in terms of the summation index z. The reason for this is that two of its numerator parameters, namely $b=-(1/2)l$ and $c=-(1/2)l+1/2$, readily leads to truncated series for $l=$ even and $l=$ odd, respectively. Then

$$\begin{aligned} J &= \mathcal{J}^{AC}_z \\ &= \pi^D\,(p^2)^{\sigma'}\,2^l\,\frac{(-i|2i+l+D)\,(-j|2j+D)}{(-i-j-D|2i+2j+l+3D)} \\ &\times {}_3F_2\left(j+D,\, -\frac{1}{2}l,\, -\frac{1}{2}l+\frac{1}{2};\right. \\ &\qquad \left. 1-i-l-D,\, 1+i+j+D \,\middle|\, 1 \right). \end{aligned} \qquad (26)$$

Now it remains for us to check the results we obtained so far by assigning explicit values for the exponents i, j, and l in (6), (10), (11), (12), and (26). Let us begin with $i=j=-1$ in (6), and in order to facilitate the comparison, we shall compute

$$2p_\mu\, I^\mu(-1,-1,D;p) = 2\,\pi^D\,(p^2)^{D-1} \times \frac{\Gamma(D)\,\Gamma(D-1)\,\Gamma(2-D)}{\Gamma(2D-1)}\,. \qquad (27)$$

This result is to be compared with the one coming from equation (26) for the particular case when $i = j = -1$, and $l = 1$. It can be seen right away that the numerator parameter $-(1/2)l + 1/2$ of the hypergeometric function ${}_3F_2$ vanishes for the particular value $l = 1$, so that only the first term in the series defining it is relevant, and

$$J(-1,-1,1,D;p) = 2\,\pi^D\,(p^2)^{D-1} \times \frac{(1|D-1)\,(1|D-2)}{(2-D|3D-3)}\,, \quad (28)$$

which is, of course, exactly equal to (27), as it should be.

From (10) we get (after contracting it with $4\,p_\mu\,p_\nu$):

$$4p_\mu\,p_\nu\,I^{\mu\nu}(-1,-1,D;p) = 4\pi^D\,(p^2)^D \times\left\{1-\frac{1}{2D}\right\}\frac{\Gamma(D+1)\,\Gamma(D-1)\,\Gamma(2-D)}{\Gamma(2D)}\,. \quad (29)$$

This is now to be compared to the result coming from (26), for the particular case when $i = j = -1$, and $l = 2$.

$$J(-1,-1,2,D;p) = 4\,\pi^D\,(p^2)^D\,\frac{(1|D)\,(1|D-2)}{(2-D|3D-2)} \times {}_2F_1\left(-1,\ -\frac{1}{2}\,;\ -D\,\middle|\ 1\right) = 4\,\pi^D\,(p^2)^D\,\frac{(1|D)\,(1|D-2)}{(2-D|3D-2)} \times\left\{1-\frac{1}{2D}\right\}. \quad (30)$$

Note that the a and f parameters coalesce into the same value $D-1$, so that the ${}_3F_2$ becomes a ${}_2F_1$ hypergeometric function. Moreover, the numerator parameter $b = -(1/2)l$ turns out to be a negative-integer unity for $l = 2$, so that the series is truncated at the second term, and the final result is exactly equal to the right-hand side (RHS) of (29).

In a completely analogous way, we get from (11), contracted with $8\,p_\mu p_\nu p_\rho$,

$$8p_\mu p_\nu p_\rho I^{\mu\nu\rho}(-1,-1,D,p) = 8\pi^D(p^2)^{D+1} \times\frac{\Gamma(D-1)\Gamma(D+2)\Gamma(2-D)}{\Gamma(2D+1)}\left\{1-\frac{3}{2(D+1)}\right\} \quad (31)$$

while from (26) with $i = j = -1$ and $l = 3$, we get

$$J(-1,-1,3,D,p) = 8\pi^D(p^2)^{D+1} \times\frac{(1|D+1)(1|D-2)}{(2-D|3D-1)}\,{}_2F_1\left(-\frac{3}{2},\,-1;\,-D-1\,\middle|\,1\right) = 8\pi^D(p^2)^{D+1}\frac{\Gamma(D-1)\Gamma(D+2)\Gamma(2-D)}{\Gamma(2D+1)} \times\left\{1-\frac{3}{2(D+1)}\right\}, \quad (32)$$

which is exactly the same as (31) above.

Finally, from (12), contracted with $16p_\mu p_\nu p_\rho p_\varsigma$, we get

$$16p_\mu p_\nu p_\rho p_\varsigma I^{\mu\nu\rho\varsigma}(-1,-1,D,p) = 16\pi^D(p^2)^{D+2}\frac{\Gamma(D-1)\Gamma(D+3)\Gamma(2-D)}{\Gamma(2D+2)} \times\left\{1-\frac{3}{D+2}+\frac{3}{4(D+1)(D+2)}\right\}, \quad (33)$$

while from (26), with $i = j = -1$ and $l = 4$, we get

$$J(-1,-1,4,D,p) = 16\pi^D(p^2)^{D+2}\frac{(1|D+2)(1|D-2)}{(2-D|3D)}\,{}_2 \times F_1\left(-2,\ -\frac{3}{2};\ -2-D\,\middle|\,1\right) = 16\pi^D(p^2)^{D+2}\frac{(1|D+2)(1|D-2)}{(2-D|3D)} \times\left\{1-\frac{3}{D+2}+\frac{3}{4(D+1)(D+2)}\right\}, \quad (34)$$

in complete agreement with (33) above.

4 Conclusion

We have shown in this paper how we can work out Feynman integrals with tensorial structures in the context of the NDIM. There are two equivalent approaches for doing this: the first is to use differential identities to work them out one by one (vector, rank-two tensor, and so on), mirroring the positive- dimensional technique; the second, to use pure NDIM methodology to get simultaneous results. The former technique does not bring any new feature, while the latter one yields this new feature of degenerate solutions, plus the bonus of having them all at once. As we have noticed before [4], the pure NDIM methodology proves more powerful in that it gives several equivalent forms of a six-fold degenerate solution for the integral. In addition, the solutions we get are *simultaneously* obtained.

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Appendix

In this appendix, we shall show in detail the equivalence of the six solutions generated by the pure NDIM technology in the computation of the one-loop Feynman integrals with tensorial structures in the numerator. In order to do this, let us first consider the solution J_z^{AC}. Its corresponding ${}_3F_2^z$ hypergeometric function has parameters given in the table of Sect. 3. It is clear from its parameter, $b_z = -(1/2)l$, that for l = even = $2m$, $m = 0,1,2,...$, the hypergeometric function is actually a truncated series. In a similar way, from its parameter $c_z = -\frac{1}{2}l + \frac{1}{2}$ for l = odd = $2m+1$, $m = 0,1,2,...$ the hypergeometric function is also a truncated series.

Truncated hypergeometric series of the form ${}_pF_q$, $p = q+1$ can be inverted [8,9] from, say, χ into χ^{-1}. For the particular case of ${}_3F_2$ with a unit argument, such inversion leads to an identity between them. This is expressed in a shorthand notation as

$$\Gamma(e-c)\Gamma(f-c)F_p(0;4,5) = (-)^m\Gamma(1-a)\Gamma(1-b) \times F_n(3;1,2) \tag{35}$$

where

$$F_p(0;4,5) = \frac{{}_3F_2(a,\, b,\, c\,;\, e,\, f\,|\,1)}{\Gamma(s)\Gamma(e)\Gamma(f)}, \tag{36}$$

and

$$F_n(3;1,2) = \frac{{}_3F_2(1+c-e,\, 1+c-f,\, c\,;\, 1-a+c,\, 1-b+c\,|\,1)}{\Gamma(s)\Gamma(1-a+c)\Gamma(1-b+c)} \tag{37}$$

with m denoting the negative-integer numerator parameter, and $s = e+f-a-b-c$.

Another way of writing this up is (for general variable χ):

$$\begin{aligned} &{}_3F_2(-m,\, \alpha_1,\, \alpha_2\,; c,\, \rho_1|\, \chi) \\ &= \mathbf{\Theta}_m\, {}_3F_2(-m,\, \beta_1,\, \beta_2\,;\, \varphi_1,\, \varphi_2\,|\,\chi^{-1}), \end{aligned} \tag{38}$$

where

$$\begin{aligned} \beta_1 &= 1-m-c \\ \beta_2 &= 1-m-\rho_1 \\ \varphi_1 &= 1-m-\alpha_1 \\ \varphi_2 &= 1-m-\alpha_2 \\ \mathbf{\Theta}_m &= \frac{(\alpha_1|m)\,(\alpha_2|m)\,(-\chi)^m}{(c|m)\,(\rho_1|m)} \end{aligned} \tag{39}$$

Let us first separate the even/odd sectors of J^z_{AC} as follows: For $l = \text{even} = 2m\,, m = 0,1,2,...,$ we define

$$\begin{aligned} {}_3F_2^{\text{even}} = {}_3F_2 \Bigg(&-m,\, -m+\frac{1}{2},\, j+D\,; \\ &1+i+j+D,\, 1-i-2m-D \,\Bigg|\, 1\Bigg), \end{aligned} \tag{40}$$

and for $l = \text{odd} = 2m+1\,, m = 0,1,2,...$ we define

$$\begin{aligned} {}_3F_2^{\text{odd}} = {}_3F_2 \Bigg(&-m,\, -m-\frac{1}{2},\, j+D\,; \\ &1+i+j+D,\, -i-2m-D \,\Bigg|\, 1\Bigg). \end{aligned} \tag{41}$$

Using equation (38) in equation (40), we get

$$\begin{aligned} {}_3F_2^{\text{even}} = \mathbf{\Upsilon}\, {}_3F_2 \Bigg(&-\frac{1}{2}l,\, -i-j-\frac{1}{2}l-D,\, i+\frac{1}{2}l+D\,; \\ &1-j-\frac{1}{2}l-D,\, \frac{1}{2} \,\Bigg|\, 1\Bigg) \end{aligned} \tag{42}$$

where

$$\begin{aligned} \mathbf{\Upsilon} &\equiv (-1)^{\frac{1}{2}l}\, \frac{(-\frac{1}{2}l+\frac{1}{2}|\frac{1}{2}l)\,(j+D|\frac{1}{2}l)}{(1+i+j+D|\frac{1}{2}l)\,(1-i-l-D|\frac{1}{2}l)} \\ &= \frac{(j+D|\frac{1}{2}l)\,(-i-j-D|-\frac{1}{2}l)\,(i+l+D|-\frac{1}{2}l)}{(\frac{1}{2}l+\frac{1}{2}|-\frac{1}{2}l)}. \end{aligned} \tag{43}$$

Plugging this into the expression of J_z, we get

$$\begin{aligned} &J^z_{AC} = 2^l\,\pi^D\,(p^2)^{\sigma'} \times \\ &\frac{\Gamma(j+\frac{1}{2}l+D)\Gamma(-i-j-\frac{1}{2}l-D)\Gamma(i+\frac{1}{2}l+D)\Gamma(\frac{1}{2}l+\frac{1}{2})}{\Gamma(-i)\Gamma(-j)\Gamma(i+j+l+D)\Gamma(\frac{1}{2})} \\ &\times {}_3F_2 \Bigg(-\frac{1}{2}l,\, -i-j-\frac{1}{2}l-D,\, i+\frac{1}{2}l+D\,; \\ &1-j-\frac{1}{2}l-D,\, \frac{1}{2} \,\Bigg|\, 1\Bigg). \end{aligned} \tag{44}$$

Using the duplication formula for the gamma function

$$\Gamma\left(\frac{1}{2}l+\frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})\Gamma(1+l)}{2^l\,\Gamma(1+\frac{1}{2}l)} \tag{45}$$

and rearranging the gamma funtions in convenient Pochhammer symbols, we arrive at the expression for $J^{y,\text{even}}_{AC}$, i.e., $J^z_{AC} = J^{y,\text{even}}_{AC}$. In a completely analogous way, starting from ${}_3F_2^{\text{odd}}$ we arrive at $J^{y,\text{odd}}_{AC}$.

Thus, when (38) is applied to our case in J^z_{AC} with $\chi = 1$, it leads us to the following conclusion: The $l = \text{even}$ sector of J^z_{AC} yields exactly the $y = \text{even}$ sector, $J^{y,\text{even}}_{AC}$, whereas the $l = \text{odd}$ sector of J^z_{AC} yields exactly the $y = \text{odd}$ sector, $J^{y,\text{odd}}_{AC}$. In order to arrive at these identities, one needs to use the duplication formula for the gamma function in the intermediate steps of the calculation.

Another identity between the ${}_3F_2$ hypergeometric functions of unity argument is given by [8,9]:

$$F_p(0;\, 4,\, 5) = F_p(0;\, 2,\, 3) \tag{46}$$

where $F_p(0;\, 4,\, 5)$ is defined in (36), and

$$F_p(0;\, 2,\, 3) = \frac{{}_3F_2(e-a,\, f-a,\, s\,;\, s+b,\, s+c\,|\,1)}{\Gamma(a)\,\Gamma(s+b)\,\Gamma(s+c)} \tag{47}$$

Plugging in the parameters of the ${}_3F_2^z$ hypergeometric function (see the table in Sect. 3) into (36), the identity (46) above yields

$$\begin{aligned} {}_3F_2^z = \mathcal{M}\, {}_3F_2 \Bigg(&1+i,\, 1-\sigma'-D,\, \frac{3}{2}-D\,; \\ &\frac{3}{2}-\frac{1}{2}l-D,\, 2-\frac{1}{2}l-D \,\Bigg|\, 1\Bigg), \end{aligned} \tag{48}$$

where $\mathcal{M}$ is a factor given by ratios of gamma functions:

$$\mathcal{M} \equiv \frac{\Gamma(\frac{3}{2}-D)\,\Gamma(1+i+j+D)\,\Gamma(1-i-l-D)}{\Gamma(j+D)\,\Gamma(\frac{3}{2}-\frac{1}{2}l-D)\,\Gamma(2-\frac{1}{2}l-D)}. \tag{49}$$

If we now redefine the ${}_3F_2$ hypergeometric function on the RHS of (48) to be our new $F_p^{\text{new}}(0;\, 4,\, 5)$, and use the fact that for terminating series, the following identity is valid [8,9],

$$\Gamma(s)\,\Gamma(e-c)\,\Gamma(f-c)\,F_p^{\rm new}(0;\,4,\,5) = \Gamma(1-a) \times\Gamma(1-f+b)\,\Gamma(1-e+b)\,F_p(1;\,0,\,2), \tag{50}$$

where

$$F_p(1;\,0,\,2) = \frac{{}_3F_2(1-a,\,1-f+b,\,1-e+b\,;\,2-s-a,\,1-a+b\,|\,1)}{\Gamma(c)\,\Gamma(2-s-a)\,\Gamma(1-a+b)}, \tag{51}$$

then we have

$$\begin{aligned} {}_3F_2^z &= \mathcal{N}\,{}_3F_2\left(-i,\,-\sigma'+\frac{1}{2}l,\,\frac{1}{2}+\frac{1}{2}l-\sigma'\,;\right.\\ &\qquad\left. 1+l-\sigma',\,1-i-\sigma'-D\,\middle|\,1\right)\\ &= \mathcal{N}\,{}_3F_2^x. \end{aligned} \tag{52}$$

We do not need to be overly concerned about the $\mathcal{N}$ factor, since both $\mathcal{M}$ and $\mathcal{N}$ are ratios of gamma functions that in the end can be rearranged conveniently to yield the desired factor present in the J_{AC}^x solution. Therefore, after some algebraic manipulation, we have $J_{AC}^z = J_{AC}^x$. In a similar manner, if we interchange parameters a and b in $F_p^{\rm new}(0;\,4,\,5)$ and proceed as above, we get $J_{AC}^z = J_{AC}^b$.

Lastly, for terminating ${}_3F_2$ hypergeometric series with parameter $c=-m$, the following identity is verified [8,9]:

$$F_p(0;\,4,\,5) = \mathbf{\Omega}\,F_n(3;\,4,\,5), \tag{53}$$

where

$$\mathbf{\Omega} \equiv (-1)^m\,\frac{\Gamma(1-a)\,\Gamma(1-b)}{\Gamma(e-c)\,\Gamma(f-c)}, \tag{54}$$

and

$$F_n(3;\,4,\,5) = \frac{{}_3F_2(1-a,\,1-b,\,s\,;\,1-a-b+e,\,1-a-b+f\,|\,1)}{\Gamma(c)\,\Gamma(1-a-c+e)\,\Gamma(1-a-b+f)}. \tag{55}$$

Substituting the parameters of the hypergeometric function ${}_3F_2^z$ in (53), we get

$$\begin{aligned} {}_3F_2^z &= \mathcal{P}\,{}_3F_2\left(1-j-D,\,1+\frac{1}{2}l,\,\frac{3}{2}-D\,;\right.\\ &\qquad\left. 2+i+\frac{1}{2}l,\,2-i-j-\frac{1}{2}l-2D\,\middle|\,1\right), \end{aligned} \tag{56}$$

where $\mathcal{P}$ is a ratio of gamma functions with which we do not concern ourselves with.

Redefining the RHS hypergeometric function in (56) as our new $F_p^{\rm new}(0;\,4,\,5)$, and using (50), we conclude that ${}_3F_2^z = \mathcal{Q}\,{}_3F_2^a$, so that, at the end, $J_{AC}^z = J_{AC}^a$. This concludes our proof of degeneracy in the solution for the Feynman integral.

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